On a family of test statistics for discretely observed
diffusion processes

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January 11, 2012

Let \( X_t, t \in [0,T] \), be a \( d \)-dimensional diffusion process solution of the following stochastic differential equation
\[
dX_t = b(\alpha, X_t)dt + \sigma(\beta, X_t)dW_t, \]
where functions \( b \) and \( \sigma \) are suitably regular and known up to the parameters \( \alpha \in \mathbb{R}^p \) and \( \beta \in \mathbb{R}^q \). The process \( X_t \) is discretely observed at times \( t_i \), such that \( t_i - t_{i-1} = \Delta_n < \infty \) for \( 1 \leq i \leq n \). The asymptotic scheme adopted in this paper is the following:
\[
T = n\Delta_n \to \infty, \quad \Delta_n \to 0 \quad \text{and} \quad n\Delta_n^2 \to 0 \quad \text{as} \quad n \to \infty.
\]
In order to test the parametric vector \( \theta = (\alpha, \beta) \) of the process \( X_t, t \in [0,T] \), this paper proposes the construction of a family of test statistics for the following hypotheses testing problem
\[
H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0.
\]

We propose a family of test statistics, related to the so called \( \phi \)-divergence measures. Let \( \phi : [0, \infty) \to \mathbb{R} \) be a convex and continuous function. Furthermore, its restriction on \( (0, \infty) \) is finite, two times continuously differentiable and such that \( \phi(1) = \phi'(1) = 0 \) and \( \phi''(1) = 1 \). We consider the following statistic
\[
D_{\phi,n}(\hat{\theta}_n, \theta_0) = \frac{1}{n} \sum_{i=1}^n \phi \left( \frac{p_i(\hat{\theta}_n)}{p_i(\theta_0)} \right),
\]
where \( p_i(\cdot) \) is a suitable approximation of the transition density of the process \( X_i \) from \( X_{t_{i-1}} \) to \( X_{t_i} \). Notice that the function \( D_{\phi,n} \) is not a true \( \phi \)-divergence, nor a proper approximation of it. Nevertheless, let \( \hat{\theta}_n \) be any consistent estimator of \( \theta \) and introduce the family of test statistics defined as follows:
\[
T_{\phi,n}(\hat{\theta}_n, \theta_0) := 2nD_{\phi,n}(\hat{\theta}_n, \theta_0).
\]
For each \( \phi \), the elements of this family are all asymptotically distribution free. In other words, our test statistics weakly converge to the chi squared distribution. In the case of contiguous alternatives, it is also possible to study in detail the power function of the tests. Extensions to other classes of processes are also considered.

Although all the tests in this family are asymptotically equivalent, we show by Monte Carlo analysis that, in the small sample case, the performance of the test depends on the choice of the function \( \phi \).